Generalizing Vizing's Theorem to Multigraphs

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Abstract

In this 501 project, we present the results of the paper "A Short Proof for a Generalization of Vizing's Theorem" by J.C. Fournier and C. Berge. Recall that Vizing's Theorem is the standard for showing that the chromatic index of a graph falls into one of two classes. In particular, for any simple graph $G, \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. For multigraphs, Vizing and Gupta showed that an upper bound of the chromatic index is $\Delta + \mu$, where μ is the maximum multiplicity of any vertex in the multigraph. The main result presented here will show that it is possible to achieve a tighter upper bound by properly coloring the edges of a graph with just $\Delta + \mu - 1$ colors. In this paper we will illustrate the proof for this result, give some examples, and present some interesting corollaries.

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Introduction

In this 501 project, we present the results of the paper "A Short Proof for a Generalization of Vizing's Theorem" by J.C. Fournier and C. Berge. [1] Recall that Vizing's Theorem is the standard for showing that the chromatic index of a graph falls into one of two classes. In particular, for any simple graph G,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

For multigraphs, Vizing and Gupta showed that an upper bound of the chromatic index is $\Delta + \mu$, where μ is the maximum multiplicity of any vertex in the multigraph. The main result presented here will show that it is possible to achieve a tighter upper bound by properly coloring the edges of a graph with just $\Delta + \mu - 1$ colors. In this paper we will illustrate the proof for this result, give some examples, and present some interesting corollaries.

Graph coloring is a subject with roots in the problems encountered by map-makers in the 1800's, but with many modern applications, from real-world scheduling and compiler optimization to sensor network design and fiber-optic communication. Let G be a graph with vertex set V and edge set E. Any edge-coloring of G is simply any assignment of colors to the edges of G. We say such an edge-coloring is proper when no two incident edges share the same color. With an unlimited number of colors, any graph can be properly edge colored. Of more interest is the minimum number of colors required to properly color the edges of a graph. We call this number the chromatic index of G, denoted $\chi'(G)$. In this paper we focus on a sharp upper bound for the chromatic index of multigraphs (graphs allowing multiple edges between vertices). We also present a some corollaries of this result.

Vadim G. Vizing (1937 – 2017) was the first to state an important bound on the chromatic index of a simple graph (a graph without multiple edges). His theorem, appropriately known as Vizing's Theorem [2], was published in 1964 and says that if G is a simple graph, then

$$\chi'(G) \leq \Delta(G) + 1. \tag{1}$$

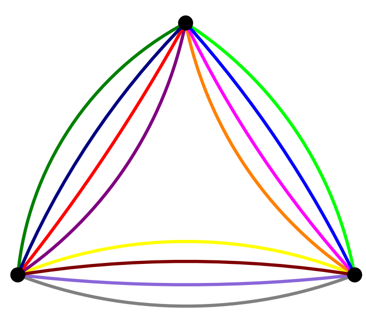
Given that any proper edge-coloring must have at least the number of colors as the maximum degree of a vertex, we have fairly tight lower and upper bounds for such edge colorings:

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1. \tag{2}$$

Building on this result for the case of multigraphs, Vizing and Gupta [3] showed that for a loopless multigraph

$$\chi'(G) \leq \Delta(G) + \mu(G) \tag{3}$$

where $\Delta(G)$ is the maximum degree of a graph G and $\mu(G)$ is the maximum multiplicity. For an example of a graph that reaches the maximum value, consider the multigraph shown below.



$$\chi'(G) = \Delta(G) + \mu(G) = 8 + 4 = 12.$$

A few years later, in 1978, Jean Claude Fournier [4] showed that if the set of vertices with maximum degree were independent, then

$$\chi'(G) = \Delta(G). \tag{4}$$

Adapting this theorem to multigraphs as Vizing and Gupta had done previously, Fournier and Berge in 1991 presented a short, but dense proof that such graphs can be properly edge colored with $\Delta(G) + \mu(G) - 1$ colors [1], so that

$$\chi'(G) \leq \Delta(G) + \mu(G) - 1. \tag{5}$$

This result improves on Vizing's and Gupta's conclusion, but it requires additional assumptions. Both methods rely on a method of "downshifting" colors from one edge to another as a key step in an induction argument. For Vizing, the configuration studied for this method was referred to as a "Vizing fan," while Fournier and Berge's adaptation of the Vizing fan coins the term "sequential f-recoloring." The main proof of this paper will illustrate the method in detail.

Main Result

Before presenting the main result, we briefly review a bit of notation. Much of this is standard and can be found in "Introduction to Graph Theory" by Douglas B. West [5].

G: a loopless multi-graph

$\Delta(G)$: the maximum degree of the graph (the maximum number of edges incident with any one vertex)

D: any upper bound for Δ(G) (useful since Δ(G) changes as we add/remove edges to the graph)

$\mu(G)$: the maximum multiplicity of a graph (the most parallel edges between any two vertices)

t: any upper bound for $\mu(G)$ (useful since $\mu(G)$ changes as we add/remove edges to the graph)

$d_G(x)$: the degree of any vertex x

$m_G(x, y)$: the number of edges between any two vertices x and y

$m_G(x) = \max_y m_G(x, y)$: the maximum multiplicity between x and any adjacent vertex.

C_y: the set of colors for the edges incident to a vertex y (relative to an edge coloring)

$\chi'(G)$: the chromatic index of G (the min number of colors needed to properly edge color G)

We now state the main result of Fournier and Berge.

Theorem [1]: Fix any positive integers D, t. Suppose G is a non-empty loopless multigraph with maximum degree $\Delta(G) \leq D$ and multiplicity $\mu(G) \leq t$. Let S be the following set:

$$S = \{x | x \in V(G); d_G(x) = D; m_G(x) = t\}.$$

If this set is independent or empty, then $\chi'(G) \leq D + t - 1$.

Before we prove this result, let's begin by unpacking what the theorem is saying. In the hypothesis, Fournier and Berge require that if there is a non-empty set of vertices which all have maximum degree D and maximum multiplicity t, then they must form an independent set for the result to hold. However, the result still holds if there are no vertices which have both maximum degree D and maximum multiplicity t. We will look at each of these cases separately. Note that if $D = \Delta + 1$ and $t = 1$, then the set is empty by construction. In this case, $D + t - 1 = \Delta + 1$, and we have Vizing's theorem (cf. [2]) for simple graphs. Also, when $D = \Delta$ and $t = 1$, the set is non-empty by construction and we obtain Fournier's result (cf. [4]) for simple graphs.

Proof of Main Result

The proof proceeds by induction on the number of edges in the graph. After establishing a base case and making a hypothesis that the theorem holds for all graphs with fewer edges than G, we show that it holds for G as well.

Base Case: Let G be a graph with one edge. Then $D \geq 1, t \geq 1$, and

$D + t - 1 \geq 1$. Clearly, a proper-edge coloring with one color exists for such a graph.

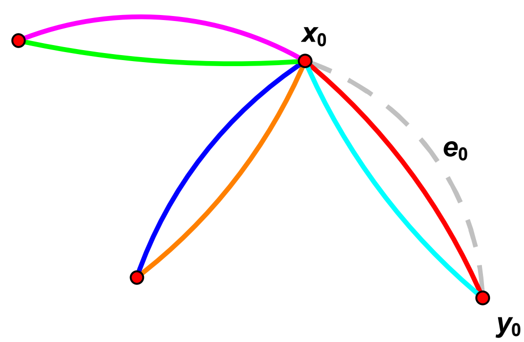
Induction Step: Let G be a non-empty graph with any number of edges ≥ 2. For any edge e of G, assume the theorem holds for $G-e$, the graph with e deleted. Note that $\Delta(G-e) \leq \Delta(G)$ and $\mu(G-e) \leq \mu(G)$ so that $\chi'(G-e) \leq D+t-1$. We want to show that

$$\chi'(G) \leq D+t-1.$$

We prove that this holds whenever the set S is independent or empty.

Case 1: $\boldsymbol{S \neq \emptyset}$

If S is not empty, then there exists a vertex $x_0 \in S$, $d_G(x_0) = D$, and $m_G(x_0) = t$. Let y_0 be any vertex adjacent to x_0 for which $m_G(x_0, y_0) = m_G(x_0)$. Since the set S is independent, we know that $d_G(y_0) < d_G(x_0)$. Let e_0 be any edge between x_0, y_0. Color the graph $G-e_0$ with $D+t-1$ colors, as guaranteed possible by the induction hypothesis. Specifically, let $g: \{1, 2, \cdots, D+t-1\}$ be a proper edge coloring of $G-e_0$. Relative to this coloring, we use C_y to denote the set of colors used on edges incident with a vertex y. We will use the induced graph by the edges between x_0 and its neighbors, illustrated below, to demonstrate the steps in the proof as we move forward.



$$\chi'(G-e_0) \leq D+t-1$$

We are going to define a sequence of edges that are incident to x_0 with the labels $e_0 = [x_0, y_0], e_1 = [x_0, y_1], \cdots, e_{k-1} = [x_0, y_{k-1}]$, where k will be determined later. For each $i \geq 1$, let the color of each edge e_i be α_i in the proper edge coloring g of $G-e_0$. We will recursively define an auxiliary function f that reserves a new color for each edge e_i in our sequence. In particular we will let $f(e_i) = \alpha_{i+1}$ where α_{i+1} is one of the $D + t - 1$ original colors used in our sequence. Note that we are not yet recoloring any of the edges, just building a sequence of edge-color pairs $\left(e_i, f(e_i)\right) = (e_i, \alpha_{i+1})$ for $i = 0, 1, \cdots, k - 1$ for some $1 \leq k \leq D$, and this sequence will comply with the following process.

A. For the edge e_0, choose any color $\alpha_1 = f(e_0)$, such that $\alpha_1 \notin C_{y_0}$. Such a color must exist since

$$\left|C_{y_0}\right| < \Delta \leq D + t - 1.$$

i	Edge	$g(e_i)$	$f(e_i)$
0	$e_0 = [x_0, y_0]$	-----	α_1

For each $i \geq 1$, repeat B until C applies.

B. If $\alpha_i \in C_{x_0}$ and if $\alpha_i \neq f(e_j)$ for all $j < i - 1$, consider the edge $e_i = [x_0, y_i]$ that is colored with α_i. Choose any color $\alpha_{i+1} = f(e_i)$ to be the color reserved for the edge e_i as long as the following two conditions hold:

1. $\alpha_{i+1} \notin C_{y_i}$ (The new color is free at y_i.)
2. $\alpha_{i+1} \neq f(e_j)$ for all $j < i$ where $y_j = y_i$ (The new color has not already been reserved for a previous edge at the same vertex.)

We know such a color α_{i+1} exists since the total number of colors q excluded by 1 and 2 is at most $\left|C_{y_i}\right| + [m_G(x_0, y_i) - 1]$. Since $y_i \sim x_0$ and S is independent, $d_G(y_i) < D$, so

$$q < \left|C_{y_i}\right| + m_G(x_0, y_i) \leq d_G(y_i) + m_G(x_0, y_i) \leq D + t - 1.$$

C. If $\alpha_k \notin C_{x_0}$, or if $\alpha_k = f(e_{j-1})$ for an index $j < k-1$, we stop and we say our recoloring sequence of edges is achieved, and the corresponding auxiliary function f is defined.

i	Edge	$g(e_i)$	$f(e_i)$
0	$e_0 = [x_0, y_0]$	-----	α_1
1	$e_1 = [x_0, y_1]$	α_1	α_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$e_i = [x_0, y_i]$	α_i	α_{i+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$e_{k-1} = [x_0, y_{k-1}]$	α_{k-1}	α_k

Note that C eventually applies, since either $\alpha_k = f(e_{j-1})$ for some $j < k-1$ or else we eventually exhaust the edges at x_0, in which case $\alpha_k \notin C_{x_0}$.

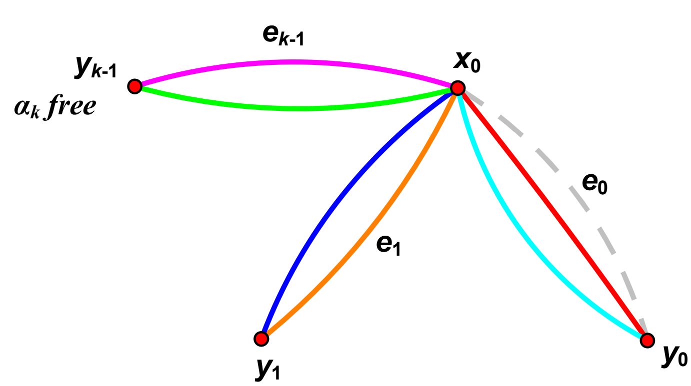
We have achieved a list of distinct edges, $e_0, e_1, \cdots, e_{k-1}$, with potential colors reserved for each. We now describe how we will eventually use this sequence to re-color G. From our list of $D + t - 1$ assume γ represents any color not assigned to C_{x_0}, and fix any edge e_i where $0 \leq i < k$. Suppose $\gamma \notin C_{y_i}$ and $\gamma \neq f(e_j)$ for any $j < i$. To sequentially f-recolor, beginning with (e_i, γ) we assign this new color γ to e_i and then "downshift", using our sequential list of f-recolorings to change the color of e_{i-1} to $f(e_{i-1})$, the color of e_{i-2} to $f(e_{i-2})$, etc. At the last step we arrive to our uncolored edge e_0 and allow $\alpha_1 = f(e_0)$. Following this process produces a proper edge-coloring of G by construction, without introducing any new colors. Our choice of γ and e_i will depend on how our process terminates. Eventually, we will run into one of the following two scenarios at C.

Scenario 1: $\alpha_k \notin C_{x_0}$ and $\alpha_k \neq f(e_{j-1})$ for $j < k-1$

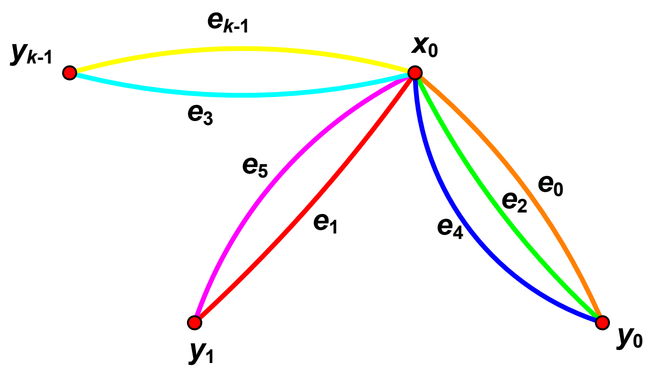
In this scenario, the color, α_k, chosen for recoloring e_{k-1} is not incident to x_0 in the original coloring g. By condition 1 in B, we know that $\alpha_k \notin C_{y_{k-1}}$. Thus, we are free to apply the sequential f-recoloring beginning with (e_{k-1}, α_k) and a valid edge coloring of G is achieved without introducing any new colors.

i	Edge	$g(e_i)$	$f(e_i)$
0	$e_0 = [x_0, y_0]$	-----	$\alpha_1 = orange$
1	$e_1 = [x_0, y_1]$	$\alpha_1 = orange$	$\alpha_2 = red$
2	$e_2 = [x_0, y_2]$	$\alpha_2 = red$	$\alpha_3 = lt\ green$
3	$e_3 = [x_0, y_3]$	$\alpha_3 = lt\ green$	$\alpha_4 = lt\ blue$
4	$e_4 = [x_0, y_4]$	$\alpha_4 = lt\ blue$	$\alpha_5 = blue$
5	$e_5 = [x_0, y_5]$	$\alpha_5 = blue$	$\alpha_6 = pink$
6	$e_6 = e_{k-1} = [x_0, y_{k-1}]$	$\alpha_{k-1} = pink$	$\alpha_k = yellow$

Original coloring of $G - e_0$:



After sequential f-recoloring:



Scenario 2: $\alpha_k = f(e_{j-1}) = \alpha_j$ for some $j < k-1$

In this scenario, the color α_k is a color originally assigned to the edge $e_j = [x_0, y_j]$ and reserved for the edge $e_{j-1} = [x_0, y_{j-1}]$. We note that $y_{k-1} \neq y_{j-1}$ by the second condition in step B. Consider a color $\beta \notin C_{x_0}$. We know such a color must exist since

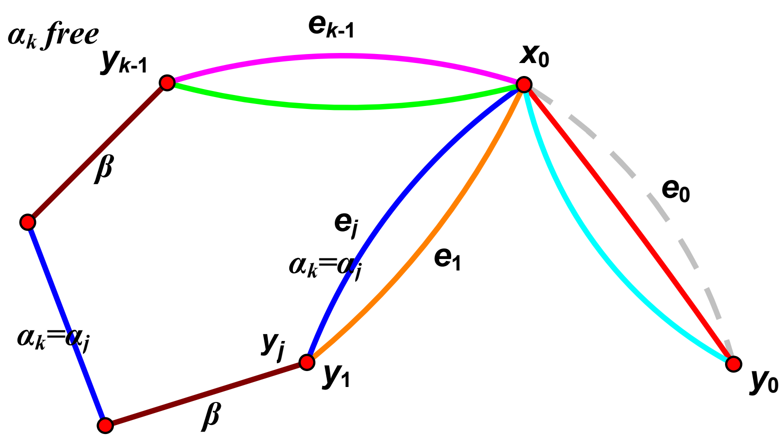
$$|C_{x_0}| < d_G(x_0) \leq \Delta \leq D + t - 1.$$

We want to consider the maximal bicolor chain of edges which we label $\langle y_{k-1}, z\rangle$ where y_{k-1} is the vertex at one endpoint of the chain and z is the vertex at the other endpoint of the chain. The two colors of the chain will be alternately β and α_k. Note that the first edge on this chain must be colored with β since $\alpha_k \notin C_{y_{k-1}}$. There are three possibilities for z that interest us. The first is when $z = x_0$. This can happen only when the chain passes through y_j and includes the edge $e_j = [x_0, y_j]$ since it has the color $\alpha_j = \alpha_k$. The second situation is when $z = y_{j-1}$. In this instance $\alpha_k \notin C_{y_{j-1}}$ and the color chain would end with the color β at this vertex. The final case of interest is when $z \neq x_0$ and $z \neq y_{j-1}$. Let g' be a new edge coloring of $G - e_0$ that swaps the colors of β and α_k in the chain $\langle y_{k-1}, z\rangle$. For any vertex y, let C'_y denote the set of colors used on edges incident with y. Consider how this new coloring affects the sequential f-recoloring for each possibility listed above.

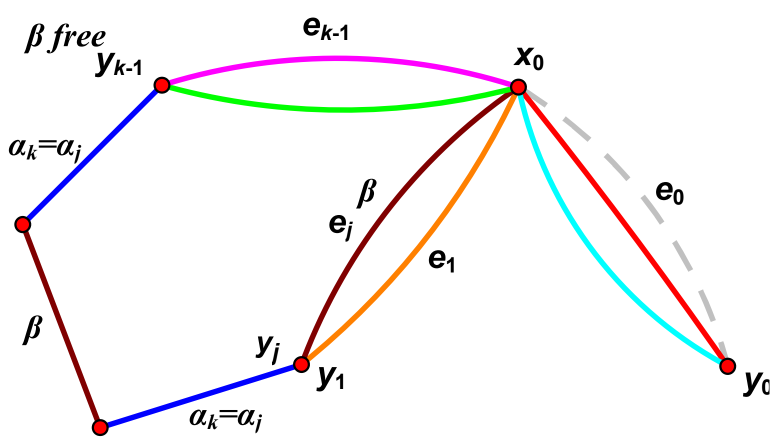
Scenario 2a: $z = x_0$

With the new coloring $\alpha_k \notin C'_{x_0}$ and the color of the edge $e_j = [x_0, y_j]$ is changed to β. With the original coloring we had that $\alpha_k \notin C_{y_{j-1}}$ and still do. Then we are able to perform the sequentially f-recoloring beginning at (e_{j-1}, α_k). This produces a valid edge coloring of G without introducing any new colors.

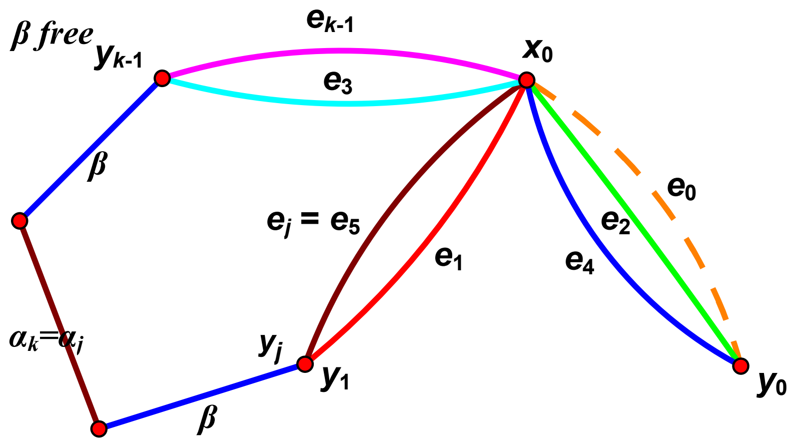
Original coloring of $G - e_0$:



Re-coloring of $\langle y_{k-1}, z\rangle$ chain:



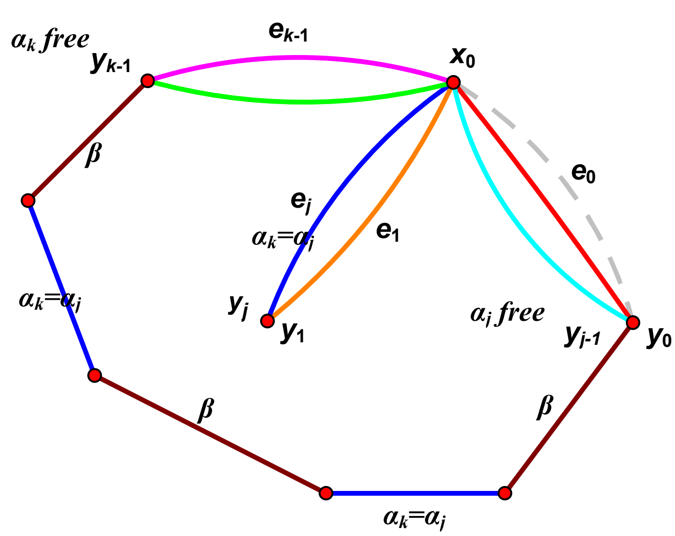
After sequential f-recoloring:



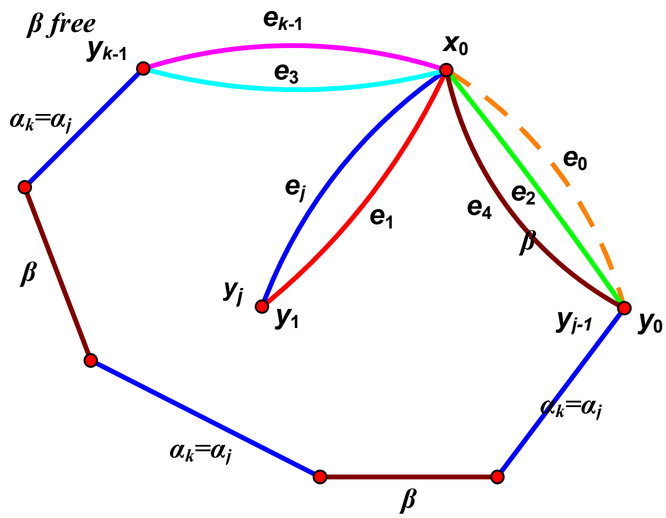
Scenario 2b: $z = y_{j-1}$

With the new coloring we have that $\beta \notin C'_{y_{j-1}}$ and $\beta' \notin C_{x_0}$. This allows us to begin the sequential f-recoloring with (e_{j-1}, β). Again, we achieve a valid edge coloring of G without introducing any new colors.

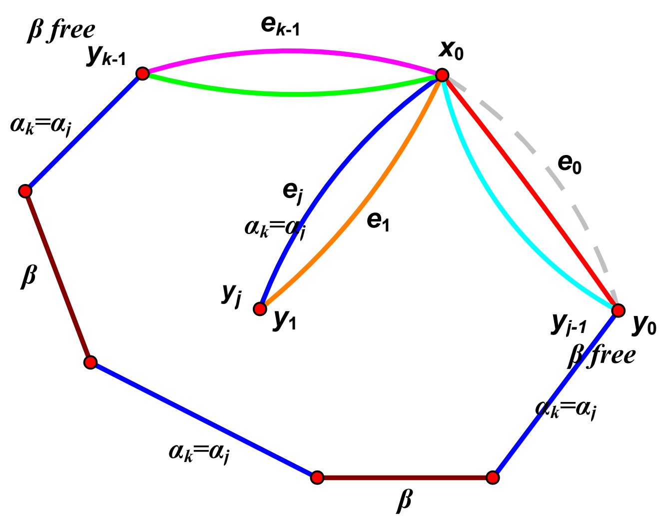
Original coloring of $G - e_0$:



Re-coloring of $\langle y_{k-1}, z\rangle$ chain:



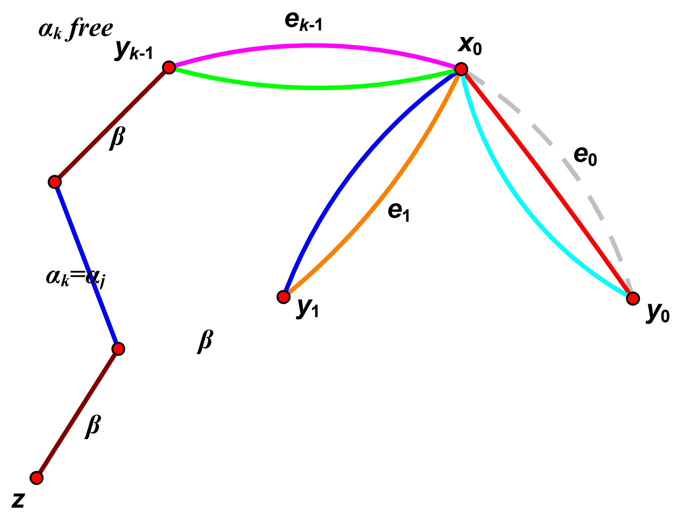
After sequential f-recoloring:



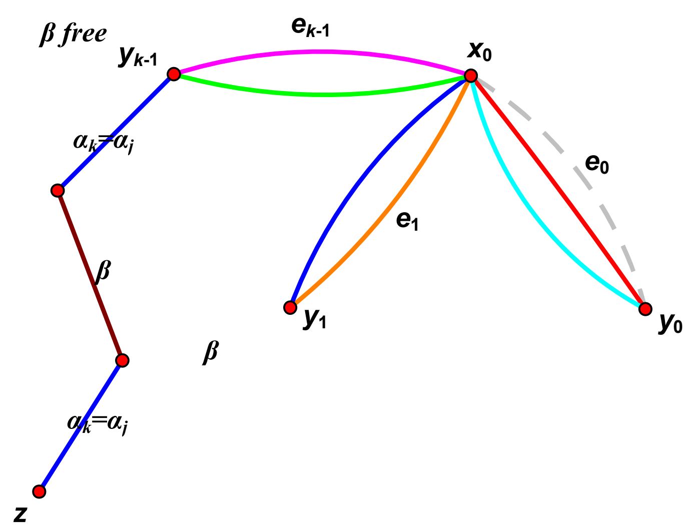
Scenario 2c: $z \neq x_0, z \neq y_{j-1}$

In this case we look to the beginning of the $\langle y_{k-1}, z\rangle$. chain. With the new coloring we have that $\beta \notin C'_{y_{k-1}}$ and $\beta \notin C'_{x_0}$. This means we can assign the color β to the edge $e_{k-1} = [x_0, y_{k-1}]$ and the sequential f-recoloring beginning with (e_{k-1}, β) produces a valid edge coloring of G without introducing any new colors.

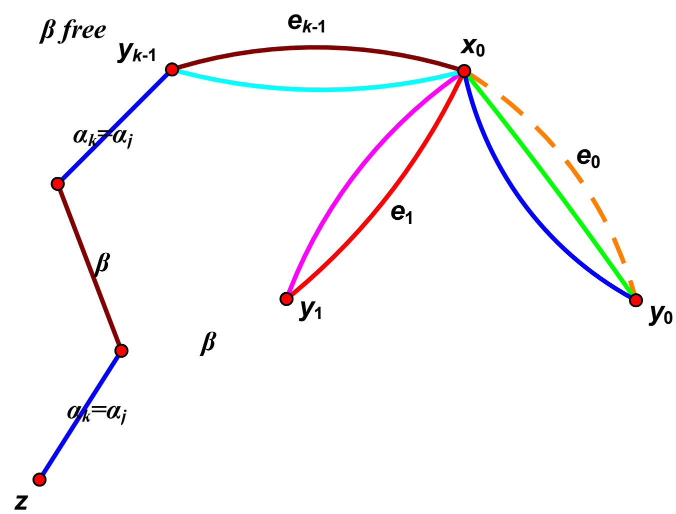
Original coloring of $G - e_0$:



Re-coloring of $\langle y_{k-1}, z\rangle$ chain:



After sequential f-recoloring



Each case yields a proper edge-coloring of G with $D + t - 1$ colors.

Case 2: $\boldsymbol{S} = \emptyset$

If S is empty, then there are no vertices which have both degree Δ and multiplicity t. In this case we choose any vertex x_0 of maximum degree such that $d_G(x_0) = D$. Let y_0 be any vertex adjacent to x_0 for which $m_G(x_0, y_0) = m_G(x_0)$. Let e_0 be any edge between x_0, y_0. Color $G - e_0$ with $D + t - 1$ colors as guaranteed possible by the induction hypothesis. Consider the same procedural steps we showed when $S \neq \emptyset$. We only need to verify that we are still able to find a free color at each vertex for our sequential f-recoloring assignments. Here is how we define f.

A. Let $\alpha_1 = f(e_0)$ such that $\alpha_1 \notin C_{y_0}$. Such a color exists since either $D > \Delta$, in which case

$$|C_{y_0}| \leq \Delta < \Delta < D + t - 1$$

or $D = \Delta$ and $m_G(x_0) < t$, in which case

$$|C_{y_0}| \leq \Delta = \Delta < D + t - 1,$$

since $t \geq 2$.

B. If $\alpha_i \in C_{x_0}$ and if $\alpha_i \neq f(e_j)$ for all $j < i - 1$, consider the edge $e_i = [x_0, y_i]$ that is colored with α_i. Choose any color $\alpha_{i+1} = f(e_i)$ to be the color reserved for the edge e_i as long as the following two conditions hold:

1. $\alpha_{i+1} \notin C_{y_i}$ (The new color is free at y_i.)
2. $\alpha_{i+1} \neq f(e_j)$ for all $j < i$ where $y_j = y_i$ (The new color has not already been reserved for a previous edge at the same vertex.)

Again, we know such a color α_{i+1} exists since the total number of colors q excluded by 1 and 2 is at most $|C_{y_i}| + [m_G(x_0, y_i) - 1]$.

$$q < |C_{y_i}| + m_G(x_0, y_i) \leq d_G(y_i) + m_G(x_0, y_i) \leq D + t - 1.$$

C. If $\alpha_k \notin C_{x_0}$, or if $\alpha_k = f(e_{j-1})$ for an index $j < k - 1$, we stop and our sequence of edges is achieved.

So far, we can still guarantee that we are able to define f for our given coloring of $G - e_0$. The last thing to check is whether there is a color β available if $\alpha_k = \alpha_j$ at some point for some $j < k - 1$. We can verify this since either $D > \Delta$, in which case

$$|C_{x_0}| = \Delta < D \leq D + t - 1,$$

or $D = \Delta$ and $m_G(x_0) < t$, in which case

$$|C_{x_0}| = \Delta = D < D + t - 1,$$

since $t \geq 2$.

Thus, when S is empty, we are able to still execute a sequential f-recoloring so that

$$\chi'(G) \leq D + t - 1.$$

QED

Corollaries

In this section, we present some interesting consequences of the main result. The first of these gives us a simpler hypothesis than the main theorem.

Corollary 1: Let G be a multigraph of maximum degree Δ and of maximum multiplicity μ. If the set of vertices of maximum degree is independent, then $\Delta + \mu - 1$ colors suffice to color the edge-set of G.

Proof: This follows immediately from the main theorem when we let $D = \Delta$ and $t = \mu$.

QED

The next corollary shows us that we can safely pre-color the edges of any maximal matching.

Corollary 2: Let G be a multigraph of maximum degree Δ and of multiplicity μ, and let M be a maximal matching of G. The edges of G can be colored with $\Delta + \mu$ colors so that all the edges in M get the same color.

Proof: Let $G - M$ be the graph obtained by removing the edges of the maximal matching M. Since M is a matching, when we remove them, we remove exactly one edge from each vertex incident to those edges. Any vertices which were not incident the edges of M are not adjacent to each other (else their shared edge would be part of M). Thus, any remaining vertices of degree Δ form an independent set (or an empty set). By the first corollary, we know that $\chi'(G - M) \leq \Delta + \mu - 1$. Color all the edges of M with one new color and we have that $\chi'(G) \leq \Delta + \mu$.

QED

Corollary 2 provides a nice algorithm for properly edge coloring a multigraph with $\Delta + \mu$ colors. Begin by finding a maximal matching M and assign all of those edges one color. Color the remaining edges with $\Delta + \mu - 1$ colors. If this proves difficult, we can find an edge $e_0 \in G - M$ such that $(G - M) - e_0$ can be properly edge colored with $\Delta + \mu - 1$ colors and then perform the sequential f-recoloring outlined in the theorem.

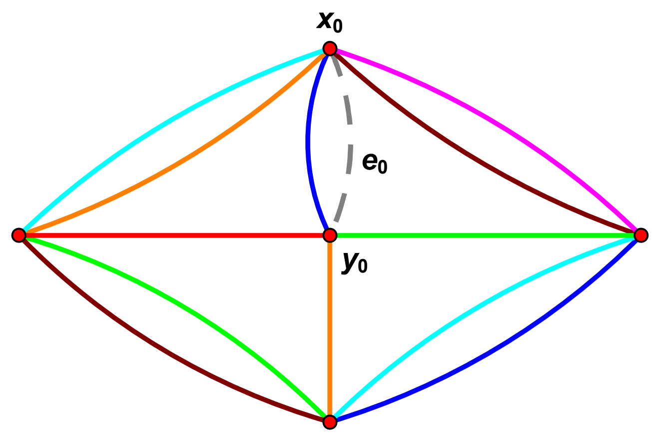
A practical application for proper edge-colorings of a graph results from Corollary 2. Suppose that the maximal matching M represents a pairing of pre-assigned matches on a given day or time slot between teams. The graph represents all the pairings that need to happen for the season or event. The chromatic index of this graph can show us how many more days or time slots would be required to achieve all pairings.

Examples

In this section we will look at an example matching Scenario 1, an example matching one of the possibilities in Scenario 2, and an example relating to Corollary 2. In each example we will begin by choosing x_0, y_0, and e_0, pre-coloring $G - e_0$, and listing $\Delta(G)$, $\mu(G)$, and the number of colors defined by our upper bound of $\Delta + \mu - 1$. After that we will show the table listing each edge in our edge set with their coloring functions $g(e_i)$ and $f(e_i)$. We will then use the appropriate scenario for carrying out the sequential f-recoloring and show the resulting graph.

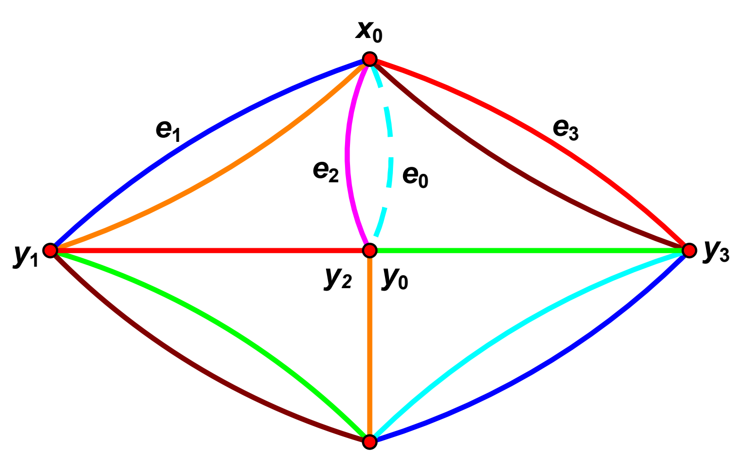
1) **Example for Scenario 1:** $\alpha_k \notin C_{x_0}$

$$\Delta(G) = 6;\ \mu(G) = 2; \Delta + u - 1 = 7$$



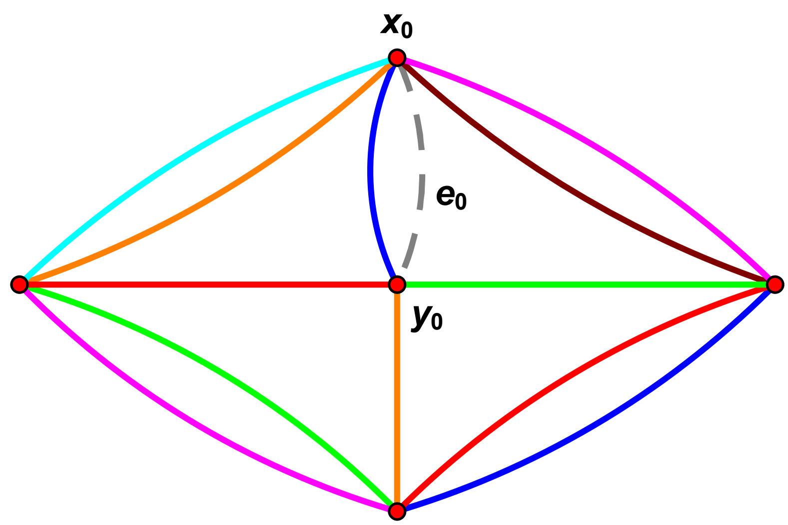
i	Edge	$g(e_i)$	$f(e_i)$
0	$e_0 = [x_0, y_0]$	-----	$\alpha_1 = lt\ blue$
1	$e_1 = [x_0, y_1]$	$\alpha_1 = lt\ blue$	$\alpha_2 = blue$
2	$e_2 = [x_0, y_2]$	$\alpha_2 = blue$	$\alpha_3 = pink$
3	$e_3 = [x_0, y_3]$	$\alpha_3 = pink$	$\alpha_4 = red$

Note that since $\alpha_4 \notin C_{x_0}$ our sequence of edges ends with $e_3 = [x_0, y_3]$. Then we are free to apply the sequential f-recoloring beginning with (e_3, α_4) and a valid edge coloring of G is achieved.



2) Example for Scenario 2: $\alpha_k = f(e_{j-1})$

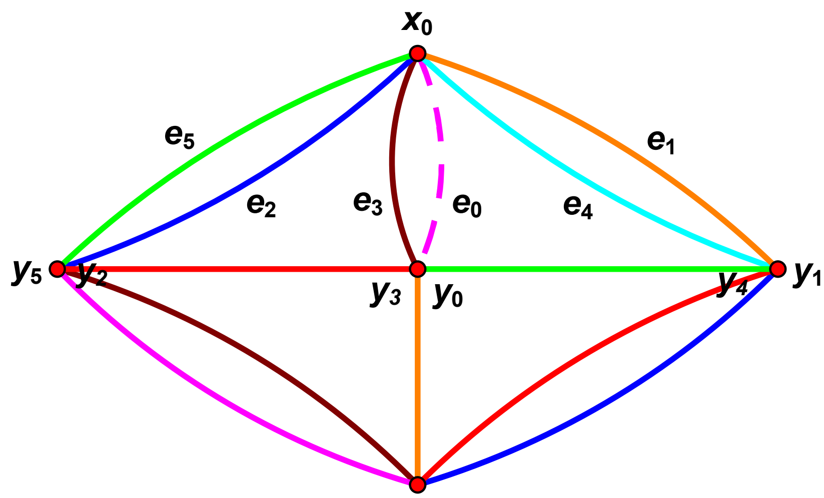
$$\Delta(G) = 6;\ \ \mu(G) = 2; \Delta + \mu - 1 = 7$$



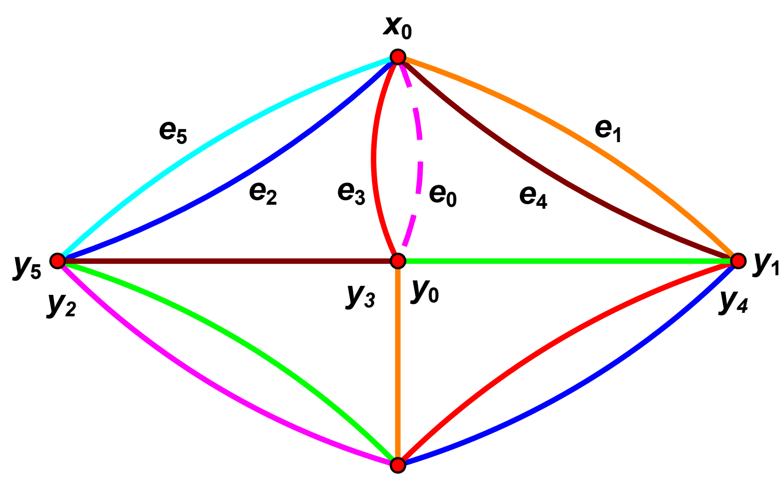
i	Edge	$g(e_i)$	$f(e_i)$
0	$e_0 = [x_0, y_0]$	-----	$\alpha_1 = pink$
1	$e_1 = [x_0, y_1]$	$\alpha_1 = pink$	$\alpha_2 = orange$
2	$e_2 = [x_0, y_2]$	$\alpha_2 = orange$	$\alpha_3 = blue$
3	$e_3 = [x_0, y_3]$	$\alpha_3 = blue$	$\alpha_4 = brown$
4	$e_4 = [x_0, y_4]$	$\alpha_4 = brown$	$\alpha_5 = lt\ blue$
5	$e_5 = [x_0, y_5]$	$\alpha_5 = lt\ blue$	$\alpha_6 = brown$

We next identify a $\langle y_{k-1}, z \rangle$ chain for including the colors $\alpha_6 = brown$ and some color $\beta \notin C_{x_0}$. The two options for β are red and green. Let's consider both options.

$\beta = green$: In this case the $\langle y_5, z\rangle$ chain is of length one and is colored green. Note that $z \neq x_0$ and $z \neq y_{j-1}$. Thus, the edge $[y_5, z]$ can be recolored $brown$ and we are free to assign the color $\beta = green$ to the edge e_5. The sequential f-recoloring beginning with $(e_5, green)$ gives a proper edge-coloring of G.



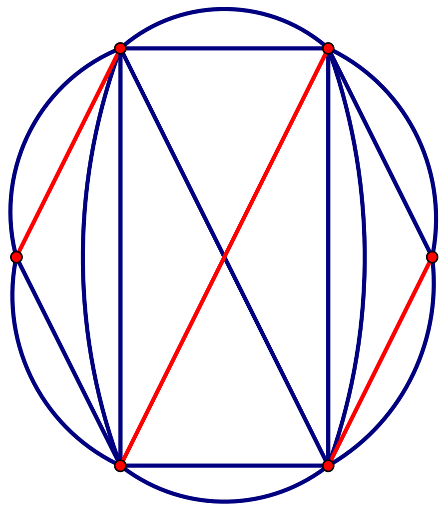
$\beta = red$: In this case the $\langle y_5, z\rangle$ chain is of length one and is colored red. Note that $z = y_{j-1} = y_3$. So we are able to assign the color $\alpha_6 = brown$ to the edge $[y_5, z]$ which makes red available at y_3 and we can sequentially f-recolor the graph beginning with (e_3, red).



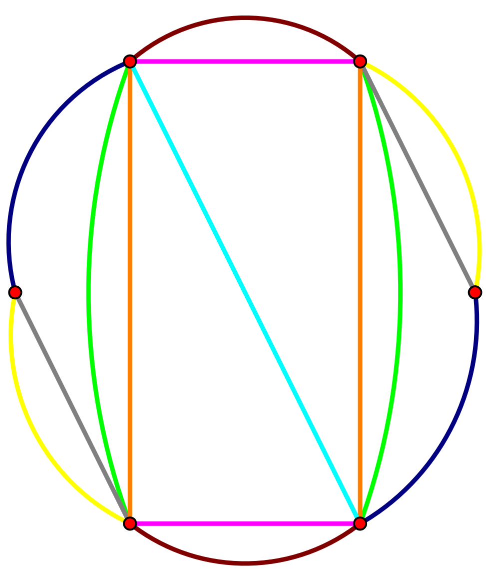
3) **Example for Corollary 2:**

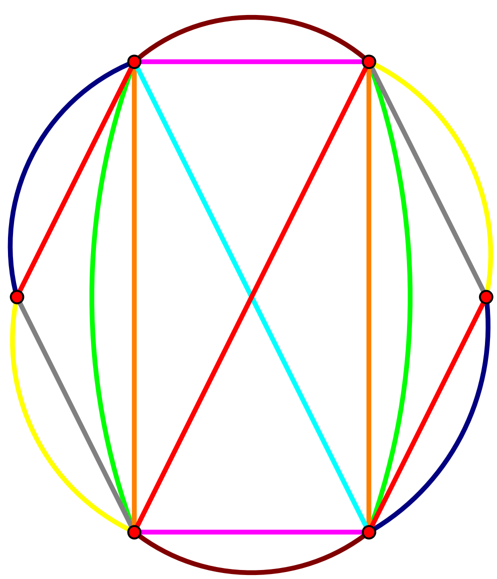
The following multigraph G has a maximal matching M pre-colored red. Corollary 2 says that we can properly edge color G with $\Delta + \mu$ colors.

$$\Delta(G) = 7;\ \ \mu(G) = 2;\ \ \Delta + \mu = 9$$



We begin by removing M from G and then using the theorem to properly edge-color the graph $G - M$ with $\Delta + \mu - 1 = 8$ colors. Below left is one such coloring. Then we simply add the pre-colored edges of M back in and G is colored with $\Delta + \mu - 1 + 1 = \Delta + \mu$ colors as desired (below right).





Conclusion

By the classical result of Vizing and Gupta (cf. [2] [3]) we know that any multigraph can be properly edge colored with $\Delta + \mu$ colors. In this work, Fournier and Berge have shown that if there is an independent set S of vertices of maximum degree Δ and maximum multiplicity μ, or if that set is empty, we can improve the chromatic index to an upper bound of $\Delta + \mu - 1$ colors. Additionally, given the restraint of a maximal matching M we can properly edge color the graph $G - M$ with $\Delta + \mu - 1$ colors and the graph G with $\Delta + \mu$ colors. Others have built upon these results. For example, in 2019, A. Girao and R.J. Kang [6] extended the result of Corollary 2 with a theorem that says if a subset M of the edge set of a multigraph has a minimum distance of 9 between its edges, and if the edges of M are arbitrarily colored from a palette of $\Delta + \mu$ colors, then there is a proper edge-coloring of G using $\Delta + \mu$ colors. We can see that the pioneering work of Vizing and Gupta, followed by the advancement of Fournier and Berge, continues to be a subject of intense research, providing further theorems and techniques which lead us to ever stronger corollaries and applications.

References

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[2] V. Vizing, "On an estimate of the chromatic class of a p-graph," *Diskretnyi Analiz,* pp. 25-30, 1964.

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[6] A. Girao and R. J. Kang, "A precolouring extension of Vizing's theorem," *Journal of Graph Theory,* vol. 92, pp. 255-260, 2019.